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Generalized Jacobian elliptic functions and their application to bifurcation problems associated with p-Laplacian ☆

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ABSTRACT

The Jacobian elliptic functions are generalized and applied to bifurcation problems associated with p-Laplacian. The values of bifurcation parameter and the corresponding solutions are represented in terms of common parameters, and a complete description of the bifurcation diagram and a closed form representation of the corresponding solutions are obtained. As a by-product of the representation, it turns out that a kind of solution is also a solution of an eigenvalue problem of $p/2$-Laplacian.

1. Introduction

In this paper we generalize the Jacobian elliptic functions and apply them to the following bifurcation problem:

$$\begin{cases} (\phi_p(u'))' + \lambda\phi_q(u)(1-|u|^q) = 0, & t \in (0,T), \\ u(0) = u(T) = 0, \end{cases} \tag{P_{pq}}$$

where $T, \lambda > 0$, $p, q > 1$ and $\phi_m(s) = |s|^{m-2}s$ $(s \neq 0)$, $= 0$ $(s = 0)$.

Problem (P_{pq}) appears frequently in various articles as stationary problems. In particular, the equation for $p = q = 2$ is called, e.g., the Allen–Cahn equation, the Chafee–Infante equation [3], and a bistable reaction–diffusion equation with logistic effect. The equation for $p = 2 < q$ is said to be a bistable reaction–diffusion equation with Allee effect. In case $p = n$ and $q = 2$ with an n-dimensional domain, an equation of this type is known as the Euler–Lagrange equation of functional related to models introduced by Ginzburg and Landau for the study of phase transitions (cf. Problem 17 in [2]).

As to (P_{pq}) for general $p > 1$, we have to mention the work [10] by Guedda and Véron. They showed that if $p = q > 1$ then there exists a positive increasing sequence $\{\lambda_n\}$ such that a pair of solutions $\pm u_n$ of (P_{pq}) with $(n-1)$-zeros $z_j = jT/n$ $(j = 1, 2, \dots, n-1)$ bifurcates from the trivial solution at $\lambda = \lambda_n$ and $|u_n| \to 1$ uniformly on any compact set of $(0,T) \setminus \{z_1, z_2, \dots, z_{n-1}\}$ as $\lambda \to \infty$. Moreover, they proved that if $p = q > 2$ then for each $n \in \mathbb{N}$ there exists $\Lambda_n > \lambda_n$ such that $\lambda > \Lambda_n$ implies $|u_n| = 1$ on *flat cores* $[z_{j-1} + \frac{T}{2n}(\frac{\Lambda_n}{\lambda})^{1/p}, z_j - \frac{T}{2n}(\frac{\Lambda_n}{\lambda})^{1/p}]$ $(j = 1, 2, \dots, n)$ of u_n, where $z_0 = 0$ and $z_n = T$. This is a great contrast to case $1 < p = q \leqslant 2$, where $|u_n| < 1$ in $[0,T]$. Since the equation in (P_{pq}) is autonomous, if u_n $(n \geqslant 2)$ has flat cores, then there exists uncountable solution with $(n-1)$-zeros near u_n, which is produced by

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expanding and contracting the flat cores with preserving its total length $T(1-(\frac{\Lambda_n}{\lambda})^{1/p})$. In this sense, the n-th branch (λ, u_n) bifurcating from $(\lambda_n, 0)$ causes the secondary bifurcation at $(\Lambda_n, u_{\Lambda_n})$ for each $n \geqslant 2$.

The phenomena of flat core in [10] above was generalized to case $p > 2$ and $q \geqslant 2$ by the author and Yamada [13]. They also studied change in bifurcation depending on the relation between p and q (as far as the first bifurcation is concerned, their proof can be applied to the case $p, q > 1$), and showed that for each $n \in \mathbb{N}$, if $p > q$ then there exists a pair of solutions $\pm u_n$ of (P_{pq}) with $(n-1)$-zeros for $\lambda > 0$; if $p = q$ then there exists $\lambda_n > 0$ such that (P_{pq}) has no solution with $(n-1)$-zeros for $\lambda \leqslant \lambda_n$ and (P_{pq}) has a pair of solutions $\pm u_n$ for $\lambda > \lambda_n$ (the same result as [10]); if $p < q$ then there exists $\lambda_n^* > 0$ such that (P_{pq}) has no solution with $(n-1)$-zeros for $\lambda < \lambda_n^*$ and (P_{pq}) has a pair of solutions $\pm u_n$ for $\lambda = \lambda_n^*$ and (P_{pq}) has two pairs of solutions $\pm u_n$, $\pm v_n$ satisfying $|u_n(t)| > |v_n(t)|$ with $t \neq z_j$ $(j = 0, 1, \ldots, n)$ for $\lambda > \lambda_n^*$. In this sense, the point $(\lambda_n^*, u_{\lambda_n^*})$ causes the *spontaneous bifurcation*. In any case, each solution u_n has flat cores for sufficiently large λ.

The purpose of this paper is to obtain a complete description of the bifurcation diagram and a closed form representation of the solutions of (P_{pq}), while the studies [10] and [13] above are done in the way of phase-plane analysis and no exact solution is given there.

For the description and representation, we first recall that the Jacobian elliptic function $\mathrm{sn}(t,k)$ with modulus $k \in [0,1)$ satisfies

$$u'' + u\bigl(1 + k^2 - 2k^2u^2\bigr) = 0 \tag{1.1}$$

(e.g., Example 4 of p. 516 in the book [15] of Whittaker and Watson) and

$$\sin t \xleftarrow[k\to+0]{} \mathrm{sn}(t,k) \xrightarrow[k\to 1-0]{} \tanh t.$$

Eq. (1.1) reminds that the solution of (P_{pq}) with $p = q = 2$ can be represented explicitly by using $\mathrm{sn}(t,k)$. Indeed, for any given $k \in (0,1)$, the value of bifurcation parameter of (P_{22}) is given by

$$\lambda_n(k) = \bigl(1 + k^2\bigr)\left(\frac{2nK(k)}{T}\right)^2 \tag{1.2}$$

for each $n \in \mathbb{N}$, with corresponding solutions $\pm u_{n,k}$, where

$$u_{n,k}(t) = \sqrt{\frac{2k^2}{1+k^2}}\,\mathrm{sn}\left(\frac{2nK(k)}{T}t, k\right) \tag{1.3}$$

and $K(k)$ is the complete elliptic integral of the first kind

$$K(k) = \int_0^1 \frac{ds}{\sqrt{(1-s^2)(1-k^2s^2)}}$$

(cf. Section 2 in [1]). Conversely, all nontrivial solutions are given by Eqs. (1.2) and (1.3), and in particular, it follows from Eq. (1.3) that all solutions satisfy $|u| < 1$.

In our study on (P_{pq}), after the fashion of Jacobi's $\mathrm{sn}(t,k)$, we introduce a new transcendental function $\mathrm{sn}_{pq}(t,k)$ with modulus $k \in [0,1)$. This satisfies

$$\bigl(\phi_p\bigl(u'\bigr)\bigr)' + \frac{q}{p^*}\phi_q(u)\bigl(1 + k^q - 2k^q|u|^q\bigr) = 0, \tag{1.4}$$

where $p^* := p/(p-1)$. Using $\mathrm{sn}_{pq}(t,k)$, we can obtain a complete description of the values of bifurcation parameter and the corresponding solutions of (P_{pq}) as Eqs. (1.2) and (1.3) with

$$K_{pq}(k) = \int_0^1 \frac{ds}{\sqrt[p]{(1-s^q)(1-k^qs^q)}}.$$

It is important that $K_{pq}(k)$ converges to $K_{\frac{p}{2},q}(0)$ as $k \to 1-0$ if and only if $p > 2$, whereas $K(k)$ diverges to ∞ as $k \to 1-0$. Indeed,

$$\lim_{k\to 1-0} K_{pq}(k) = \int_0^1 \frac{ds}{(1-s^q)^{\frac{2}{p}}} = K_{\frac{p}{2},q}(0).$$

Similarly, we can find

$$\sin_{pq}(t) \xleftarrow[k\to+0]{} \mathrm{sn}_{pq}(t,k) \xrightarrow[k\to 1-0]{} \sin_{\frac{p}{2},q}(t),$$

where $\sin_{pq}(t)$ is a generalized trigonometric function, introduced by Drábek and Manásevich [8] (see Section 2). These convergence properties yield the existence of special solutions, not necessarily $|u| < 1$, and we can really construct the solutions of (P_{pq}) with flat cores. Moreover, $\sin_{\frac{p}{2},q}(t)$ satisfies Eq. (1.4) with $k = 0$ and p replaced by $p/2$ as well as Eq. (1.4) with $k = 1$. Thus, we obtain the following (curious) property: a kind of solution of (P_{pq}) is also a solution of the nonlinear eigenvalue problem with $p/2$-Laplacian

$$\begin{cases} \left(\phi_{\frac{p}{2}}\left(u'\right)\right)' + \lambda\phi_q(u) = 0, \quad t \in (0, T), \\ u(0) = u(T) = 0. \end{cases}$$

This paper is organized as follows. In Section 2, we review the generalized trigonometric function $\sin_{pq}(t)$ and its properties, given by Drábek and Manásevich [8]. In Section 3, we define the new transcendental function $\mathrm{sn}_{pq}(t, k)$, which is a generalization of the Jacobian elliptic function $\mathrm{sn}(t, k)$ and an extension of $\sin_{pq}(t)$ in the sense that $\mathrm{sn}_{pq}(t, 0) = \sin_{pq}(t)$, similarly to the fact $\mathrm{sn}(t, 0) = \sin t$. In Section 4, we apply them to the bifurcation problem (P_{pq}), considered in [10] and [13], and obtain complete descriptions of the values of bifurcation parameter and the corresponding solutions.

2. A result of Drábek and Manásevich

Our results are closely related to generalized trigonometric functions, introduced by Drábek and Manásevich [8] (see also [7]). In this section, we review their definitions and properties.

For $\sigma \in [0, 1]$, we define

$$\operatorname{arcsin}_{pq}(\sigma) := \int_0^\sigma \frac{ds}{(1 - s^q)^{\frac{1}{p}}},$$

where $p, q > 1$. Letting $s = z^{1/q}$, we have

$$\operatorname{arcsin}_{pq}(\sigma) = \frac{1}{q} \int_0^{\sigma^q} z^{\frac{1}{q}-1}(1 - z)^{-\frac{1}{p}}\, dz = \frac{1}{q} \tilde{B}\left(\frac{1}{q}, \frac{1}{p^*}, \sigma^q\right),$$

where $\tilde{B}(s, t, u)$ denotes the incomplete beta function

$$\tilde{B}(s, t, u) = \int_0^u z^{s-1}(1 - z)^{t-1}\, dz.$$

We define the constant π_{pq} as

$$\pi_{pq} := 2 \operatorname{arcsin}_{pq}(1) = \frac{2}{q} B\left(\frac{1}{q}, \frac{1}{p^*}\right),$$

where $B(s, t)$ denotes the beta function

$$B(s, t) = \tilde{B}(s, t, 1) = \int_0^1 z^{s-1}(1 - z)^{t-1}\, dz.$$

We have that $\operatorname{arcsin}_{pq} : [0, 1] \to [0, \pi_{pq}/2]$, and is strictly increasing. Let us denote its inverse by $\sin_{pq}$. Then, $\sin_{pq} : [0, \pi_{pq}/2] \to [0, 1]$ and is strictly increasing. We extend $\sin_{pq}$ to all $\mathbb{R}$ (and still denote this extension by $\sin_{pq}$) in the following form: for $t \in [\pi_{pq}/2, \pi_{pq}]$, we set $\sin_{pq}(t) := \sin_{pq}(\pi_{pq} - t)$, then for $t \in [-\pi_{pq}, 0]$, we define $\sin_{pq}(t) := -\sin_{pq}(-t)$, and finally we extend $\sin_{pq}$ to all $\mathbb{R}$ as a $2\pi_{pq}$ periodic function.

Remark 2.1. We immediately find that $\sin_{22}(t) = \sin(t)$ and $\pi_{22} = \pi$ from the properties of the beta function. Moreover, $\sin_{pp}(t) = \sin_p(t)$ and $\pi_{pp} = \pi_p = \frac{2\pi}{p \sin\frac{\pi}{p}}$, where $\sin_p$ and π_p are the generalized sine function and its half-period, respectively, introduced by Elbert [9] (see also [4–7]).

We immediately obtain for all $\mathbb{R}$

$$\left(\phi_p\left(u'\right)\right)' + \frac{q}{p^*}\phi_q(u) = 0. \tag{2.1}$$

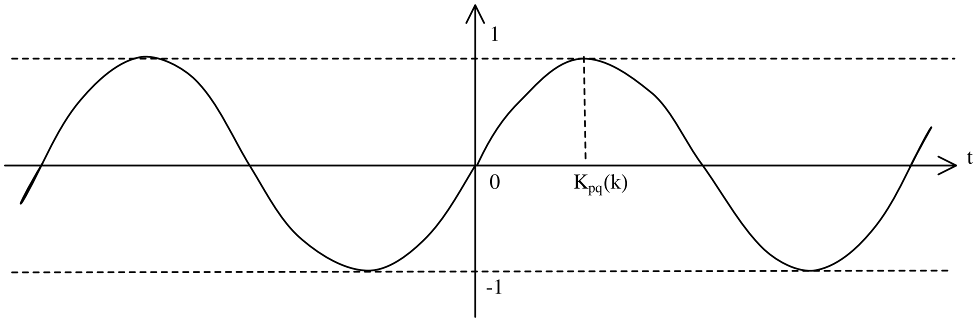

Fig. 1. $\mathrm{sn}_{pq}(t,k)$.

Thanks to the generalized function, we can represent solutions of the following (p,q)-eigenvalue problem: for $T,\lambda>0$ and $p,q>1$

$$\begin{cases} (\phi_p(u'))' + \lambda\phi_q(u) = 0, & t \in (0,T), \\ u(0) = u(T) = 0. \end{cases} \tag{E_{pq}}$$

Problem (E_{pq}) has been studied by many authors. In particular, in paper [12] of Ôtani, the existence of infinitely many multi-node solutions was proved by using subdifferential operators method and phase-plane analysis combined with symmetry properties of the solutions. After that, Drábek and Manásevich [8] provided explicit forms of the whole spectrum and the corresponding eigenfunctions for (E_{pq}) (see also [4] and [7]), that is,

Theorem 2.1. *(See [8].) All nontrivial solutions of* (E_{pq}) *are given as follows. For any given $R>0$, the set of eigenvalues of* (E_{pq}) *is given by*

$$\lambda_n(R) = \frac{q}{p^*}\left(\frac{n\pi_{pq}}{T}\right)^p R^{p-q}$$

for each $n\in\mathbb{N}$, with corresponding eigenfunctions $\pm u_{n,R}$, where

$$u_{n,R}(t) = R\sin_{pq}\left(\frac{n\pi_{pq}}{T}t\right). \tag{2.2}$$

3. Generalized Jacobian elliptic functions

In this section, we shall introduce new transcendental functions, which generalize the Jacobian elliptic functions.

For $\sigma\in[0,1]$ and $k\in[0,1)$, we define

$$\operatorname{arcsn}_{pq}(\sigma) = \operatorname{arcsn}_{pq}(\sigma,k) := \int_0^\sigma \frac{ds}{\sqrt[p]{(1-s^q)(1-k^q s^q)}}, \tag{3.1}$$

where $p,q>1$. We define the constant $K_{pq}(k)$ as

$$K_{pq} = K_{pq}(k) := \operatorname{arcsn}_{pq}(1,k) = \int_0^1 \frac{ds}{\sqrt[p]{(1-s^q)(1-k^q s^q)}}.$$

We have that $\operatorname{arcsn}_{pq}:[0,1]\to[0,K_{pq}]$, and is strictly increasing. Let us denote its inverse by $\mathrm{sn}_{pq}(\cdot)=\mathrm{sn}_{pq}(\cdot,k)$. Then, $\mathrm{sn}_{pq}:[0,K_{pq}]\to[0,1]$ and is strictly increasing. We extend sn_{pq} to all $\mathbb{R}$ (and still denote this extension by sn_{pq}) in the following form: for $t\in[K_{pq},2K_{pq}]$, we set $\mathrm{sn}_{pq}(t):=\mathrm{sn}_{pq}(2K_{pq}-t)$, then for $t\in[-2K_{pq},0]$, we define $\mathrm{sn}_{pq}(t):=-\mathrm{sn}_{pq}(-t)$, and finally we extend sn_{pq} to all $\mathbb{R}$ as a $4K_{pq}$ periodic function. See Fig. 1.

The following proposition is crucial to our study.

Proposition 3.1. *For $p,q>1$, K_{pq} is continuous and strictly increasing in $[0,1)$, $2K_{pq}(0)=\pi_{pq}$ and $\mathrm{sn}_{pq}(t,0)=\sin_{pq}(t)$. Moreover,*

$$\lim_{k\to1-0} 2K_{pq}(k) = \begin{cases} \pi_{\frac{p}{2},q} & \text{if } p>2, \\ \infty & \text{if } 1<p\leqslant 2, \end{cases}$$
$$\lim_{k\to1-0} \mathrm{sn}_{pq}(t,k) = \sin_{\frac{p}{2},q}(t) \quad \text{if } p>2.$$

Proof. The first half is trivial from the definitions of K_{pq} and sn_{pq}. If $p > 2$, then the monotone convergence theorem of Beppo Levi gives

$$\lim_{k\to 1-0} 2K_{pq}(k) = 2\int_0^1 \frac{ds}{(1-s^q)^{\frac{2}{p}}} = 2\arcsin_{\frac{p}{2},q}(1) = \pi_{\frac{p}{2},q}.$$

If $1 < p \leqslant 2$, then $2K_{pq}(k)$ diverges to ∞ as $k \to 1-0$ by Fatou's lemma.

The last property is proved as follows. By the symmetry of $\mathrm{sn}_{pq}(\cdot,k)$, we may assume $t > 0$. Suppose $p > 2$ and that there exist $t_0, \varepsilon > 0$ and $\{k_j\}$ such that $k_j \to 1$ as $j \to \infty$ and

$$\left|\sigma_{k_j} - \sin_{\frac{p}{2},q}(t_0)\right| \geqslant \varepsilon, \tag{3.2}$$

where $\sigma_{k_j} = \mathrm{sn}_{pq}(t_0, k_j)$. Let $n \in \mathbb{Z}$ be the number satisfying $t_0 \in I_n := [n\pi_{\frac{p}{2},q}/2, (n+1)\pi_{\frac{p}{2},q}/2)$ and $j \in \mathbb{N}$ a large number satisfying $t_0 \in I_n(k_j) := [nK_{pq}(k_j), (n+1)K_{pq}(k_j))$. We write $\mathrm{sn}_{pq}^{(n)}(\cdot,k_j)$ as $\mathrm{sn}_{pq}(\cdot,k_j)$ on $I_n(k_j)$ and $\sin_{pq}^{(n)}(\cdot)$ as $\sin_{pq}(\cdot)$ on I_n. Now, since $\sigma_{k_j} = sn_{pq}^{(n)}(t_0,k_j)$ is bounded, we can choose a subsequence $\{k_{j'}\}$ of $\{k_j\}$ such that $\sigma_{k_{j'}} \to \sigma$ for some $\sigma \in [-1,1]$ as $j' \to \infty$. Thus, as $j' \to \infty$

$$t_0 = nK_{pq}(k_{j'}) + \mathrm{arcsn}_{pq}(\sigma_{k_{j'}}) \to \frac{n\pi_{\frac{p}{2},q}}{2} + \arcsin_{\frac{p}{2},q}(\sigma),$$

and hence $\sigma = \sin^{(n)}_{\frac{p}{2},q}(t_0)$, which contradicts (3.2). □

Remark 3.1. In case $p > 2$, $2K_{pq}(k)$ and $\mathrm{sn}_{pq}(t,k)$ converge to the finite value $\pi_{\frac{p}{2},q}$ and to the finite-periodic function $\sin_{\frac{p}{2},q}(t)$ as $k \to 1-0$, respectively. This is quite different from case $p = 2$, where $2K_{2q}(k)$ diverges to ∞ and $\mathrm{sn}_{22}(t,k)$ converges to the monotone increasing function $\tanh t$ as $k \to 1-0$.

We define for $t \in [0, K_{pq}]$

$$\mathrm{cn}_{pq}(t) := \left(1 - \mathrm{sn}_{pq}^q(t)\right)^{\frac{1}{p}},$$
$$\mathrm{dn}_{pq}(t) := \left(1 - k^q\,\mathrm{sn}_{pq}^q(t)\right)^{\frac{1}{p}},$$

then we obtain

$$\mathrm{cn}_{pq}^p(t) + \mathrm{sn}_{pq}^q(t) = 1,$$
$$\frac{d}{dt}\mathrm{sn}_{pq}(t) = \mathrm{cn}_{pq}(t)\,\mathrm{dn}_{pq}(t).$$

Proposition 3.2. *For $p, q > 1$, sn_{pq} satisfies for all $\mathbb{R}$*

$$\left(\phi_p\left(u'\right)\right)' + \frac{q}{p^*}\phi_q(u)\left(1 + k^q - 2k^q|u|^q\right) = 0, \tag{3.3}$$

which includes Eq. (2.1) *as case $k = 0$.*

Proof. For $t \in (0, K_{pq}(k))$ we have

$$\begin{aligned}
\left(\phi_p\left(u'\right)\right)' &= \left(\phi_p\left(\mathrm{cn}_{pq}(t)\,\mathrm{dn}_{pq}(t)\right)\right)' \\
&= \left(\left(\left(1 - \mathrm{sn}_{pq}^q(t)\right)\left(1 - k^q\,\mathrm{sn}_{pq}^q(t)\right)\right)^{\frac{1}{p^*}}\right)' \\
&= \frac{1}{p^*}\left(\left(1 - \mathrm{sn}_{pq}^q(t)\right)\left(1 - k^q\,\mathrm{sn}_{pq}^q(t)\right)\right)^{-\frac{1}{p}}\left(-q\,\mathrm{sn}_{pq}^{q-1}(t)\cdot\left(1 + k^q - 2k^q\,\mathrm{sn}_{pq}^q(t)\right)\right)\cdot \mathrm{cn}_{pq}(t)\,\mathrm{dn}_{pq}(t) \\
&= -\frac{q}{p^*}\phi_q(u)\left(1 + k^q - 2k^q u^q\right).
\end{aligned}$$

By symmetry of sn_{pq}, Eq. (3.3) holds true for $t \neq t_n := nK_{pq}(k)$, $n \in \mathbb{Z}$. Since $\lim_{t\to t_n}(\phi_p(u'))'$ exists, $\phi_p(u')$ is differentiable also at $t = t_n$ and satisfies Eq. (3.3) for all $\mathbb{R}$ in the classical sense. □

Remark 3.2. Letting $s = \sin_{pq}(t)$ in Eq. (3.1), we have

$$\operatorname{arcsn}_{pq}(\sigma, k) = \int_0^{\operatorname{arcsin}_{pq}(\sigma)} \frac{dt}{\sqrt[p]{1 - k^q \sin_{pq}^q(t)}}.$$

We define the amplitude function $\operatorname{am}_{pq}(\cdot, k) : [0, K_{pq}(k)] \to [0, \pi_{pq}/2]$ by

$$t = \int_0^{\operatorname{am}_{pq}(t,k)} \frac{d\theta}{\sqrt[p]{1 - k^q \sin_{pq}^q(\theta)}},$$

thus sn_{pq} is represented by $\sin_{pq}$ as

$$\operatorname{sn}_{pq}(t, k) = \sin_{pq}\bigl(\operatorname{am}_{pq}(t, k)\bigr).$$

4. Application

Let $T, \lambda > 0$ and $p, q > 1$. We consider the bifurcation problem

$$\begin{cases} \bigl(\phi_p(u')\bigr)' + \lambda \phi_q(u)\bigl(1 - |u|^q\bigr) = 0, & t \in (0, T), \\ u(0) = u(T) = 0. \end{cases} \tag{P_{pq}}$$

Problem (P_{pq}) has been studied by Berger and Fraenkel [1] and Chafee and Infante [3] $(p = q = 2)$, Wang and Kazarinoff [14] and Korman, Li and Ouyang [11] $(p = 2 < q)$, Guedda and Véron [10] $(p = q > 1)$, and Takeuchi and Yamada [13] $(p > 2, q \geqslant 2)$. However, there is no study providing explicit forms of the values of bifurcation parameter and the corresponding solutions for (P_{pq}).

We follow closely the ideas of [8]. It will be convenient to find first the solution to the initial value problem

$$\begin{cases} \bigl(\phi_p(u')\bigr)' + \lambda \phi_q(u)\bigl(1 - |u|^q\bigr) = 0, \\ u(0) = 0, \quad u'(0) = \alpha, \end{cases} \tag{4.1}$$

where without loss of generality we may assume $\alpha > 0$.

Let u be a solution to Eq. (4.1) and let $t(\alpha)$ be the first zero point of $u'(t)$. On interval $(0, t(\alpha))$, u satisfies $u(t) > 0$ and $u'(t) > 0$, and thus

$$\frac{u'(t)^p}{p^*} + \lambda \frac{F(u)}{q} = \lambda \frac{F(R)}{q} = \frac{\alpha^p}{p^*},$$

where $F(s) = s^q - \frac{1}{2}s^{2q}$ and $R = u(t(\alpha))$. Since we are interested in functions satisfying the boundary condition of (P_{pq}), it suffices to assume $0 < R \leqslant 1$, which means $|u| \leqslant 1$. Moreover, we restrict to $0 < R < 1$ and concentrate solutions satisfying $|u| < 1$ for a while.

Solving for u' and integrating, we find

$$\left(\frac{q}{\lambda p^*}\right)^{\frac{1}{p}} \int_0^t \frac{u'(s)}{\sqrt[p]{F(R) - F(u(s))}}\, ds = t,$$

which after a change of variable can be written as

$$t = \left(\frac{q}{\lambda p^*}\right)^{\frac{1}{p}} \int_0^{\frac{u(t)}{R}} \frac{R}{\sqrt[p]{F(R) - F(Rs)}}\, ds. \tag{4.2}$$

It is easy to verify that

$$F(R) - F(Rs) = F(R)\bigl(1 - s^q\bigr)\left(1 - \frac{R^q}{2 - R^q} s^q\right),$$

and hence

$$t=\left(\frac{q}{\lambda p^*}\right)^{\frac{1}{p}}\frac{R}{F(R)^{\frac{1}{p}}}\int_0^{\frac{u(t)}{R}}\frac{ds}{\sqrt[p]{(1-s^q)(1-k^q s^q)}}\quad\left(k^q:=\frac{R^q}{2-R^q}\right)$$

$$=\left(\frac{q}{\lambda p^*}\right)^{\frac{1}{p}}\frac{R}{F(R)^{\frac{1}{p}}}\operatorname{arcsn}_{pq}\left(\frac{u(t)}{R},k\right).$$

Then we obtain that the solution to Eq. (4.1) can be written as

$$u(t)=R\operatorname{sn}_{pq}\left(\left(\frac{\lambda p^*}{q}\right)^{\frac{1}{p}}\frac{F(R)^{\frac{1}{p}}}{R}t,k\right),\tag{4.3}$$

where

$$k=\left(\frac{R^q}{2-R^q}\right)^{\frac{1}{q}}.\tag{4.4}$$

We first observe the structure of the set of all nontrivial solutions of (P_{pq}) satisfying $|u|<1$.

Theorem 4.1 *($|u|<1$). All nontrivial solutions of (P_{pq}) for $p\in(1,2]$ and all nontrivial solutions of (P_{pq}) with $|u|<1$ for $p>2$ are given as follows. For any given $k\in(0,1)$, the value of bifurcation parameter λ of (P_{pq}) is given by*

$$\lambda_n(k)=\frac{q}{p^*}\left(1+k^q\right)\left(\frac{2k^q}{1+k^q}\right)^{\frac{p}{q}-1}\left(\frac{2nK_{pq}(k)}{T}\right)^p\tag{4.5}$$

for each $n\in\mathbb{N}$, with corresponding solutions $\pm u_{n,k}$, where

$$u_{n,k}(t)=\left(\frac{2k^q}{1+k^q}\right)^{\frac{1}{q}}\operatorname{sn}_{pq}\left(\frac{2nK_{pq}(k)}{T}t,k\right).\tag{4.6}$$

Proof. For $k\in(0,1)$ given, we impose that function (4.3) with $R\in(0,1)$, where R is uniquely decided from Eq. (4.4), satisfies the boundary conditions in (P_{pq}). Then, we obtain

$$\left(\frac{\lambda p^*}{q}\right)^{\frac{1}{p}}\frac{F(R)^{\frac{1}{p}}}{R}T=2nK_{pq}(k),\quad n\in\mathbb{N},$$

where from Eq. (4.4)

$$\frac{F(R)^{\frac{1}{p}}}{R}=\left(\frac{2k^q}{1+k^q}\right)^{\frac{1}{p}-\frac{1}{q}}\left(1+k^q\right)^{-\frac{1}{p}}.$$

Thus λ is given by Eq. (4.5). Expression (4.6) for the solutions follows then directly from Eq. (4.3).

It remains to show that no other nontrivial solution of (P_{pq}) is obtained when $1<p\leqslant 2$. Assume the contrary. Then there exist $t_*>0$ and a nontrivial solution u of (P_{pq}) with $R=u(t_*)=1$. However, the right-hand side of Eq. (4.2) with $t=t_*$ diverges because $\sqrt[p]{F(1)-F(s)}=O((1-s^q)^{\frac{2}{p}})$ as $s\to 1-0$. Thus, $t_*=\infty$, which is a contradiction. $\square$

Next we find solutions of (P_{pq}) with $|u|\leqslant 1$, except the solutions given by Theorem 4.1. Now we assume $p>2$. From Proposition 3.1, one of solutions of Eq. (4.1) can be obtained by $k\to 1-0$ in Eq. (4.3) with Eq. (4.4), namely

$$u(t)=\sin_{\frac{p}{2},q}\left(\left(\frac{\lambda p^*}{2q}\right)^{\frac{1}{p}}t\right).$$

We take a number t_* as $(\frac{\lambda p^*}{2q})^{\frac{1}{p}}t_*=\pi_{\frac{p}{2},q}/2$, then u attains 1 at $t=t_*$ (note that t_* is well defined if and only if $p>2$). Using this u, we can make the other solutions of Eq. (4.1) as follows. In the phase-plane, the orbit $(u(t),u'(t))$ arrives at the equilibrium point $(1,0)$ at $t=t_*$ and can stay there for any finite time τ before it begins to leave there. Then, the interval $[t_*,t_*+\tau]$ is a flat core of the solution. Similarly, there is the other equilibrium point $(-1,0)$, where the orbit can stay, and the solution has another flat core of any finite length. Thus we have solutions of Eq. (4.1) attaining ± 1 with any number of flat cores.

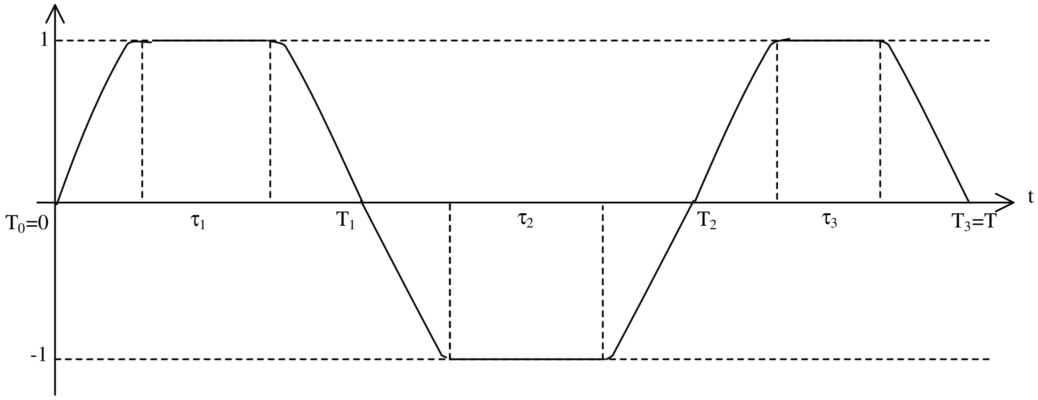

Fig. 2. $u_{3,\{\tau_i\}}$ for some $\{\tau_i\}$, an element of $U_{3,\tau}$.

Theorem 4.2 *($|u| \leqslant 1$). Let $p > 2$, then all nontrivial solutions of (P_{pq}) without $|u| < 1$ (that is, $|u|$ attains 1) are given as follows. For any given $\tau \in [0, T)$, the value of bifurcation parameter λ of (P_{pq}) is given by*

$$\Lambda_n(\tau) = \frac{2q}{p^*}\left(\frac{n\pi_{\frac{p}{2},q}}{T-\tau}\right)^p$$

for each $n \in \mathbb{N}$, with corresponding solutions $\pm u_{n,\{\tau_i\}}$, where $u_{n,\{\tau_i\}}$ is any function given as follows: for any $\{\tau_i\}_{i=1}^n$ with $\tau_i \geqslant 0$ and $\sum_{i=1}^n \tau_i = \tau$

$$u_{n,\{\tau_i\}}(t) = \begin{cases} (-1)^{j-1}\sin_{\frac{p}{2},q}\left(\frac{n\pi_{\frac{p}{2},q}}{T-\tau}(t-T_{j-1})\right) & \text{if } T_{j-1} \leqslant t \leqslant T_{j-1} + \frac{T-\tau}{2n}, \\ (-1)^{j-1} & \text{if } T_{j-1} + \frac{T-\tau}{2n} \leqslant t \leqslant T_j - \frac{T-\tau}{2n}, \\ (-1)^{j-1}\sin_{\frac{p}{2},q}\left(\frac{n\pi_{\frac{p}{2},q}}{T-\tau}(T_j - t)\right) & \text{if } T_j - \frac{T-\tau}{2n} \leqslant t \leqslant T_j, \\ j = 1, 2, \ldots, n, \end{cases} \tag{4.7}$$

where $T_0 = 0$ and $T_j = \frac{(T-\tau)j}{n} + \sum_{i=1}^j \tau_i$ for $j = 1, 2, \ldots, n$. See Fig. 2.

Proof. For each $n \in \mathbb{N}$, it suffices to construct solutions with $(n-1)$-zeros. Let $\tau \in [0, T)$. They are all generated by the value of bifurcation parameter and the corresponding solution of (P_{pq}) with T replaced by $T - \tau$, namely,

$$\Lambda_n(\tau) = \frac{2q}{p^*}\left(\frac{n\pi_{\frac{p}{2},q}}{T-\tau}\right)^p,$$
$$u_{n,\tau}(t) = \sin_{\frac{p}{2},q}\left(\frac{n\pi_{\frac{p}{2},q}}{T-\tau}t\right),$$

which are obtained from Eqs. (4.5) and (4.6) with $k \to 1-0$, respectively. In the phase-plane, the orbit $(u_{n,\tau}(t), u'_{n,\tau}(t))$ goes through the equilibrium points $(\pm 1, 0)$ in n-times without staying there as t increases from 0 to $T - \tau$. Therefore, if the orbit stays the i-th equilibrium point for time τ_i, where $\tau_1 + \tau_2 + \cdots + \tau_n = \tau$, then we can obtain solution (4.7) with n-flat cores in $[0, T]$. $\square$

In Theorems 4.1 and 4.2, we gave parameters k and τ to obtain the value of bifurcation parameter and the corresponding solution of (P_{pq}). Conversely, giving any $\lambda > 0$, we can observe the set S_λ of all solutions of (P_{pq}) by considering the inverses of λ_n and Λ_n. For any $\tau \in [0, T)$, we denote by $\pm U_{n,\tau}$ the sets of solutions of (P_{pq}) with n-flat cores of total length τ:

$$\pm U_{n,\tau} = \left\{\pm u_{n,\{\tau_i\}}\colon \tau_i \geqslant 0 \text{ and } \sum_{i=1}^n \tau_i = \tau\right\}.$$

Theorem 4.3. *Let $p > 1$ and $q > 1$.*

Case $p > q$. For any $\lambda > 0$ there exists a strictly decreasing positive sequence $\{k_j\}_{j=1}^\infty$ such that $k_j \to 0$ as $j \to \infty$ and (see Fig. 3)

$$S_\lambda = \{0\} \cup \bigcup_{j=1}^\infty \{\pm u_{j,k_j}\}.$$

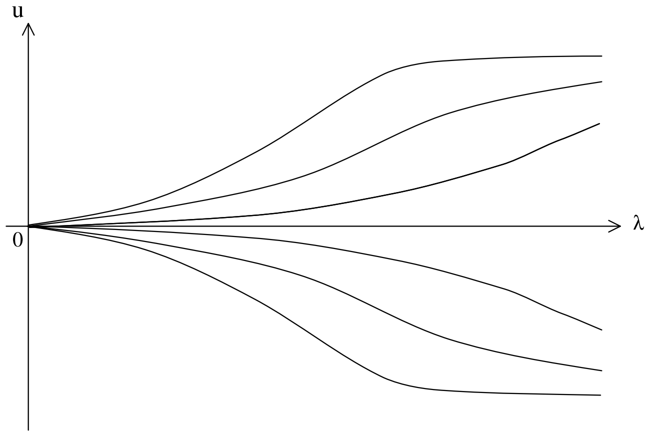

Fig. 3. Bifurcation diagram for $p > q$.

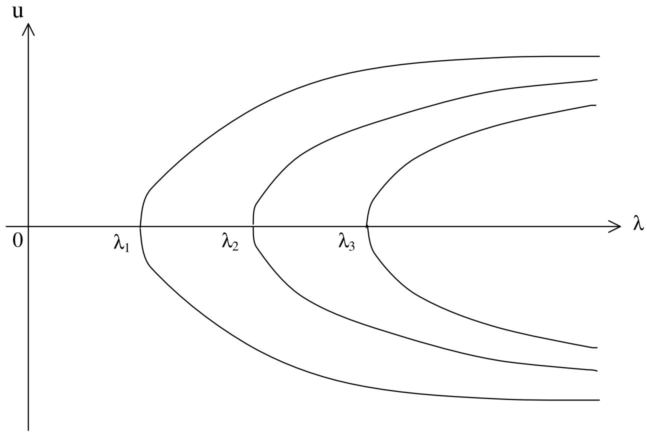

Fig. 4. Bifurcation diagram for $p = q$, where $\lambda_n = (p-1)(\frac{n\pi_p}{T})^p$.

Case $p = q$. *Let* $\lambda_n = (p-1)(n\pi_p/T)^p$ *with* $\pi_p = 2\pi/(p\sin(\pi/p))$. *If* $0 < \lambda \leqslant \lambda_1$, *then* $S_\lambda = \{0\}$. *If* $\lambda_n < \lambda \leqslant \lambda_{n+1}$, $n \in \mathbb{N}$, *then there exists a strictly decreasing positive sequence* $\{k_j\}_{j=1}^n$ *such that* (*see Fig.* 4)

$$S_\lambda = \{0\} \cup \bigcup_{j=1}^{n} \{\pm u_{j,k_j}\}.$$

Case $p < q$. *There exists* $\lambda_* > 0$ *such that if* $0 < \lambda < \lambda_*$, *then* $S_\lambda = \{0\}$. *If* $n^p\lambda_* \leqslant \lambda < (n+1)^p\lambda_*$, $n \in \mathbb{N}$, *then there exist a strictly decreasing positive sequence* $\{k_j\}_{j=1}^n$ *and a strictly increasing positive sequence* $\{\ell_j\}_{j=1}^n$ *such that* $k_j > \ell_j$, $j = 1, 2, \ldots, n-1$ *and* (*see Fig.* 5)

$$S_\lambda = \{0\} \cup \bigcup_{j=1}^{n} \{\pm u_{j,k_j}\} \cup \bigcup_{j=1}^{n} \{\pm u_{j,\ell_j}\},$$

where $u_{n,k_n} = u_{n,\ell_n}$ *with* $k_n = \ell_n$ *for* $\lambda = n^p\lambda_1$ *and* $|u_{n,k_n}| > |u_{n,\ell_n}|$ $(t \neq jT/n,\ j = 1, 2, \ldots, n-1)$ *with* $k_n > \ell_n$ *otherwise.*

In any case, each k_j, ℓ_j *are calculated by Eq.* (4.5) *for* λ *and* j, *and the corresponding solution is given in form* (4.6).

When $1 < p \leqslant 2$, *we have* $k_j < 1$ (*i.e.*, $\pm u_{j,k_j}$ *have no flat core*). *When* $p > 2$, *in addition, if*

$$\lambda \geqslant \frac{2q}{p^*}\left(\frac{m\pi_{\frac{p}{2},q}}{T}\right)^p, \quad m \in \mathbb{N}, \tag{4.8}$$

then for each $j = 1, 2, \ldots, m$, *the set* $\{\pm u_{j,k_j}\}$ *above is replaced by* $\pm U_{j,\tau}$, *where*

$$\tau = T - j\pi_{\frac{p}{2},q}\left(\frac{2q}{\lambda p^*}\right)^{\frac{1}{p}}. \tag{4.9}$$

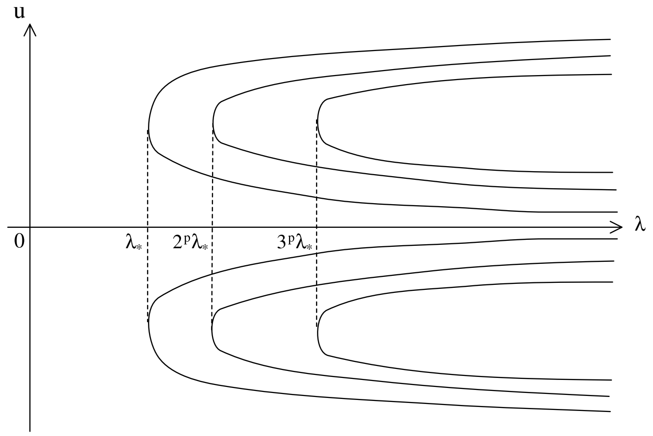

Fig. 5. Bifurcation diagram for $p < q$.

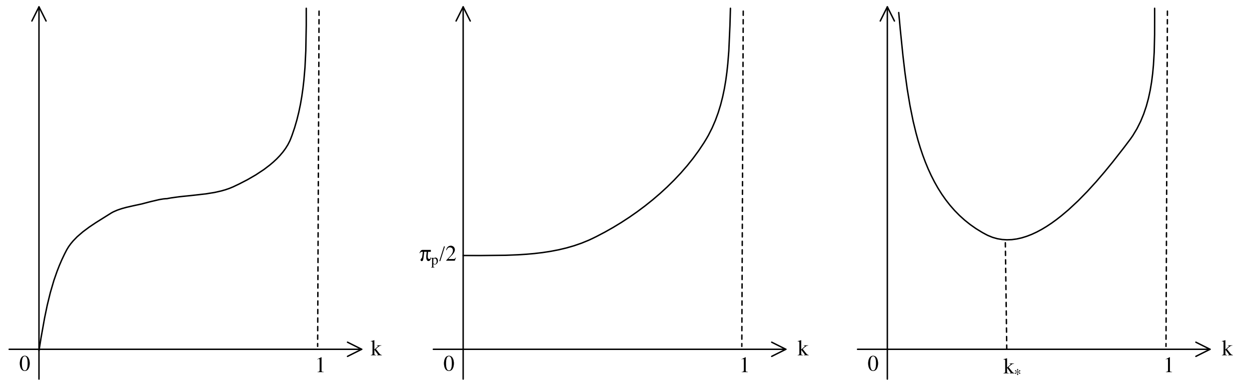

Fig. 6. $\Phi(k)$ for $p > q$, $p = q$ and $p < q$ in case $1 < p \leqslant 2$.

Proof. First we assume $1 < p \leqslant 2$. In this case, we have already known that all nontrivial solutions of (P_{pq}) are obtained by Theorem 4.1.

Now we fix $\lambda > 0$. We obtain that λ is the j-th smallest value for which (P_{pq}) has a solution if and only if from Eq. (4.5) there exists $k \in (0,1)$ such that $\lambda = \lambda_j(k)$, that is,

$$\Phi(k) = c(\lambda), \tag{4.10}$$

where

$$\Phi(k) = \left(1+k^q\right)^{\frac{1}{p}} \left(\frac{2k^q}{1+k^q}\right)^{\frac{1}{q}-\frac{1}{p}} K_{pq}(k),$$
$$c(\lambda) = \frac{T}{2j}\left(\frac{\lambda p^*}{q}\right)^{\frac{1}{p}}.$$

Case $p > q$. $\Phi(k)$ is strictly increasing in $(0,1)$ and it follows from Proposition 3.1 that $\Phi(0) = 0$ and $\lim_{k\to 1-0}\Phi(k) = \infty$. Thus, there exists a unique $k = k_j(\lambda)$ satisfying Eq. (4.10). For j and k_j, a unique solution u_{j,k_j} of (P_{pq}) is obtained by Eq. (4.6). See Fig. 6.

Case $p = q$. $\Phi(k)$ is strictly increasing in $(0,1)$ and it follows from Proposition 3.1 that $\Phi(0) = \pi_p/2$ and $\lim_{k\to 1-0}\Phi(k) = \infty$. Thus, if $c(\lambda) > \pi_p/2$, namely, $\lambda > \lambda_j$, then there exists a unique $k = k_j(\lambda)$ satisfying Eq. (4.10). For j and k_j, a unique solution u_{j,k_j} of (P_{pq}) is obtained by Eq. (4.6). See Fig. 6.

Case $p < q$. It is clear that $\lim_{k\to +0}\Phi(k) = \lim_{k\to 1-0}\Phi(k) = \infty$. Changing variable $r = \frac{k^q}{1+k^q}$, we can write Φ as

$$\Psi(r) = \int_0^1 \frac{(1+s^q)^{\frac{1}{p}-\frac{1}{q}}}{(1-s^q)^{\frac{1}{p}}}\,\psi\big((1+s^q)r\big)\,ds, \quad r \in (0,1/2),$$

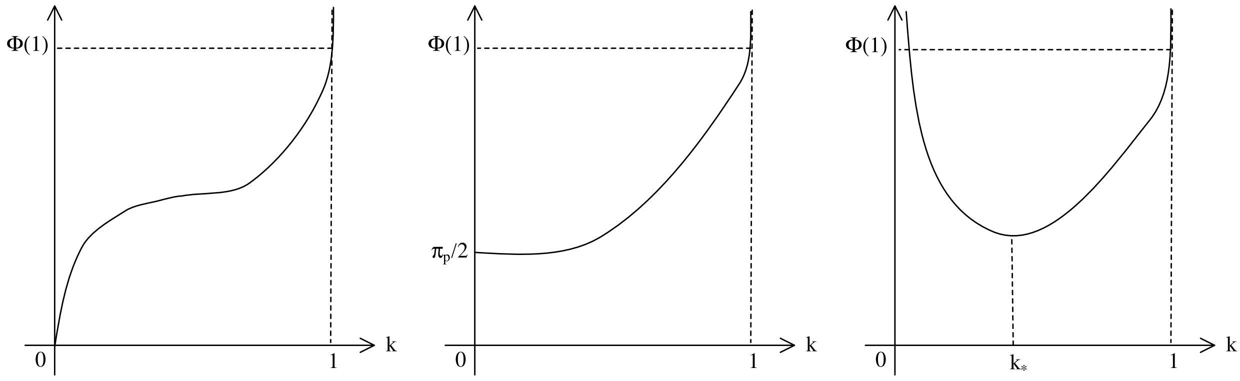

Fig. 7. $\Phi(k)$ for $p>q$, $p=q$ and $p<q$ in case $p>2$, where $\Phi(1)=2^{\frac{1}{p}-1}\pi_{\frac{p}{2},q}$.

where $\psi(t)=(2t)^{\frac{1}{q}-\frac{1}{p}}(1-t)^{-\frac{1}{p}}$. It is easy to see that ψ is convex in $(0,1)$ because $\psi(t)>0$ and

$$\bigl(\log\psi(t)\bigr)''=\left(\frac{1}{p}-\frac{1}{q}\right)\frac{1}{t^2}+\frac{1}{p}\frac{1}{(1-t)^2}>0.$$

Then, Ψ is twice-differentiable in $(0,1/2)$ and

$$\Psi''(r)=\int_0^1\frac{(1+s^q)^{\frac{1}{p}-\frac{1}{q}+2}}{(1-s^q)^{\frac{1}{p}}}\psi''\bigl((1+s^q)r\bigr)\,ds>0.$$

Thus, Ψ is convex and there exists $k_*\in(0,1)$ such that $\Phi(k_*)$ is the only one critical value, and hence the minimum of Φ in $(0,1)$.

If $c(\lambda)=\Phi(k_*)$, namely, $\lambda=j^p\lambda_*$, where $\lambda_*=(2\Phi(k_*)/T)^p q/p^*$, then k_* satisfies Eq. (4.10). For j and k_*, a unique solution u_{j,k_*} of (P_{pq}) is obtained by Eq. (4.6). If $c(\lambda)>\Phi(k_*)$, namely, $\lambda>j^p\lambda_*$, then there exist $k=k_j(\lambda)$ and $\ell_j(\lambda)$ such that $k_j(\lambda)=\Phi^{-1}(c(\lambda))\in(k_*,1)$, $\ell_j(\lambda)=\Phi^{-1}(c(\lambda))\in(0,k_*)$. For j, k_j and ℓ_j, solutions u_{j,k_j} and u_{j,ℓ_j} of (P_{pq}) are obtained by Eq. (4.6). See Fig. 6.

Next, we assume $p>2$. In any case, a similar proof as above with $\lim_{k\to1-0}\Phi(k)=2^{\frac{1}{p}-1}\pi_{\frac{p}{2},q}$ instead of $\lim_{k\to1-0}\Phi(k)=\infty$ implies that it is impossible to find $k_m\in(0,1)$ above satisfying Eq. (4.10), provided λ satisfies (4.8). Then, however, for each $j=1,2,\dots,m$, we can take $\tau\in[0,T)$ as (4.9) so that $\lambda=\Lambda_j(\tau)$. Therefore, Theorem 4.2 yields the solutions $u_{j,\{\tau_i\}}$, where $\{\tau_i\}_{i=1}^j$ is any sequence satisfying that $\tau_i\geqslant0$, $\sum_{i=1}^j\tau_i=\tau$. See Fig. 7. $\square$

It follows directly from representation (4.7) of Theorem 4.2 that a kind of solution of (P_{pq}) with p-Laplacian is also a solution of $(\mathrm{E}_{\frac{p}{2},q})$, an eigenfunction of $p/2$-Laplacian.

Corollary 4.1. *Let $p>2$. For each $n\in\mathbb{N}$ and $\tau\in[0,T)$, any function in $\pm U_{n,\tau}$ satisfies*

$$\bigl(\phi_{\frac{p}{2}}(u')\bigr)'+\frac{(p-2)q}{p}\left(\frac{n\pi_{\frac{p}{2},q}}{T-\tau}\right)^{\frac{p}{2}}\phi_q(u)=0$$

in the intervals where $|u|<1$. In particular, for each $n\in\mathbb{N}$, each function in $\pm U_{n,0}$ is a solution of $(\mathrm{E}_{\frac{p}{2},q})$, that is,

$$\begin{cases}\bigl(\phi_{\frac{p}{2}}(u')\bigr)'+\dfrac{(p-2)q}{p}\left(\dfrac{n\pi_{\frac{p}{2},q}}{T}\right)^{\frac{p}{2}}\phi_q(u)=0, & t\in(0,T),\\ u(0)=u(T)=0,\end{cases}$$

and hence, the solution is characterized by $\pm u_{n,R}$ with $R=1$ in solution (2.2) with p replaced by $p/2$.

Remark 4.1. The principle of reduction in p in Corollary 4.1 can be also explained formally as follows. We recall that $u=\sin_{\frac{p}{2},q}(t)$ satisfies Eq. (1.4) with $k=1$, that is,

$$\bigl(\phi_p(u')\bigr)'+\frac{2q}{p^*}\phi_q(u)\bigl(1-|u|^q\bigr)=0$$

and $1-|u|^q=|u'|^{\frac{p}{2}}$. Thus, formal calculation with the aid of $(\phi_p(u'))'=(p-1)|u'|^{p-2}u''$ allows us to derive

$$\left(\phi_{\frac{p}{2}}\left(u'\right)\right)'+\frac{q}{(\frac{p}{2})^*}\phi_q(u)=0.$$

Acknowledgments

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